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LETTER TO THE EDITOR

The eigenvalue spectrum of a large symmetric random matrix with a finite mean

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Abstract. A recently published Letter by Kota and Potbhare obtains the averaged spectrum of a large symmetric random matrix each element of which has a finite mean: their results disagree with two recent calculations which predict that under certain circumstances a single isolated eigenvalue splits off from the continuous semicircular distribution of eigenvalues associated with the random part of the matrix. This Letter offers a simple re-derivation of this result and corrects the error in the work of Kota and Potbhare.

It is well known (Wigner 1958) that if a large $N\times N$ real symmetric matrix has elements which are normally distributed with mean zero and variance J^2/N (but whose diagonal elements have variance $2J^2/N$), it possesses an averaged eigenvalue spectrum given by the semicircular distribution

$$\rho_0(\lambda)=\begin{cases}\dfrac{1}{2\pi J^2}(4J^2-\lambda^2)^{1/2} & |\lambda|<2J\\[2ex] 0 & |\lambda|>2J.\end{cases}\tag{1}$$

The statistical properties of such a matrix are invariant under an orthogonal transformation and the collection of all different matrices with these properties is referred to as the Gaussian orthogonal ensemble (GOE).

Edwards and Jones (1976) used the $n\to 0$ method to obtain an asymptotic expression for the ensemble averaged spectrum of a large symmetric $N\times N$ real matrix $\mathbf{M}$ each of whose elements was normally distributed as in the Gaussian orthogonal ensemble but with mean M_0/N instead of zero. They deduced a spectrum of the form

$$\rho(\lambda)=\begin{cases}\rho_0(\lambda)+N^{-1}\delta\left[\lambda-\left(M_0+\dfrac{J^2}{M_0}\right)\right] & |M_0|>J\\[2ex] \rho_0(\lambda) & |M_0|<J.\end{cases}\tag{2}$$

The same result was given by Kosterlitz *et al* (1976a, to be referred to as KTJ). They used the standard techniques of Slater–Koster localised perturbation theory and pointed out that the mode at $\lambda=M_0+J^2/M_0$ was akin to the localised state associated with a single impurity in a crystal.

Kota and Potbhare (1977) have recently claimed that the result (2) is incorrect and that the spectrum should in fact be

$$\tilde{\rho}(\lambda)=\rho_0(\lambda)+N^{-1}\rho_0(\lambda-M_0). \tag{3}$$

Both expressions, (2) and (3) lead, of course, to the correct answer for the spectrum of the degenerate matrix in which $J\to 0$.

Since no details of their calculation were given in KTJ we provide these here and point out the error in the calculation by Kota and Potbhare.

Let $\mathbf{M}=\mathbf{J}+\mathbf{V}$ where $\mathbf{J}$ is a member of the Gaussian orthogonal ensemble and $V_{ij}=M_0/N$. A straightforward orthogonal transformation on $\mathbf{M}$ will leave invariant the statistical properties of $\mathbf{J}$ and can be chosen so that

$$\mathbf{V}\to\mathbf{M}_0=\begin{pmatrix} M_0 & 0 \\ & \\ 0 & 0\end{pmatrix}$$

It thus suffices to study the ensemble averaged spectrum of the matrix

$$\tilde{\mathbf{M}}=\mathbf{J}+\mathbf{M}_0. \tag{4}$$

Denote by $|\alpha\rangle$ an eigenvector of $\mathbf{J}$ with eigenvalue J_α: then we have $|i\rangle=\Sigma_\alpha\, u_\alpha(i)|\alpha\rangle$ as the definition of an orthogonal transformation to this new representation. In this new basis $\tilde{\mathbf{M}}$ has matrix elements:

$$\tilde{M}_{\alpha\beta}=J_\alpha\delta_{\alpha\beta}+M_0u_\alpha(1)u_\beta(1). \tag{5}$$

The eigenvectors of $\tilde{\mathbf{M}}$ are given by the equation

$$J_\alpha a_\alpha+M_0u_\alpha(1)\sum_\beta u_\beta(1)a_\beta=\lambda a_\alpha \tag{6}$$

which has non-trivial solutions if

$$f(\lambda)\equiv M_0\sum_\alpha(\lambda-J_\alpha)^{-1}u_\alpha^2(1)=1. \tag{7}$$

If $M_0>J$, this equation has $(N-1)$ solutions in the range of the eigenvalues of $\mathbf{J}$, and a single solution outside this range, whilst if $M_0<J$ all solutions lie within the range. Defining the Green functions $\mathbf{G}$ and $\mathbf{G}^0$ by $\mathbf{G}=(\mathbf{I}\lambda-\mathbf{J}-\mathbf{M}_0)^{-1}$ and $\mathbf{G}^0=(\mathbf{I}\lambda-\mathbf{J})^{-1}$ respectively, we see that $G_{11}^0\equiv f(\lambda)/M_0$ and (if λ is given the usual small positive imaginary part) the density of eigenvalues can be written as $-(N\pi)^{-1}\,\mathrm{Im\,Tr}\,\mathbf{G}$. The problem is now very similar to that of a single localised impurity in a crystal, the role of the impurity perturbation being taken by the single non-zero matrix element M_0. The standard perturbation techniques for a single impurity (Koster and Slater 1954) can now be used and yield the exact results

$$G_{ii}=G_{ii}^0+\sum_{n=1}^{\infty}M_0^n(G_{11}^0)^{n-1}G_{i1}^0G_{1i}^0. \tag{8}$$

From which we find that

$$\mathrm{Tr}\,\mathbf{G}=\mathrm{Tr}\,\mathbf{G}^0-(1-f(\lambda))^{-1}\partial f/\partial\lambda. \tag{9}$$

It is known that the components $u_\alpha(1)$ which appear in (7) are normally distributed with mean zero and variance N^{-1} (Porter and Rosenweig 1960, Kosterlitz *et al*

1976b). We now argue that a sum such as $\Sigma_\alpha u_\alpha^2(1)(\lambda - J_\alpha)^{-1}$ may be replaced by an integral of the form $\int_{-2J}^{+2J} \rho_0(q)(\lambda - q)^{-1}\, dq$ where $\rho_0(\lambda)$ is the semicircular distribution (1). This procedure is correct to order N^{-1} provided λ is a distance greater than $O(1/N)$ from the cut $-2J < \lambda < 2J$. This is equivalent to treating (8) as a perturbation expansion which is averaged term by term: similar arguments using the statistical properties of the components $u_\alpha(1)$ and retaining only terms of leading order in N^{-1} at each order of the perturbation expansion leads us to replace the averages of products of the G_{11}^0 by the product of the averages and gives a final result

$$\mathrm{Tr}\,\langle \mathbf{G}\rangle = \mathrm{Tr}\,\langle \mathbf{G}^0\rangle - \frac{M_0 \partial\langle G_{11}^0\rangle/\partial\lambda}{1 - M_0\langle G_{11}^0\rangle} \tag{10}$$

where

$$\langle G_{11}^0\rangle = \int_{-2J}^{+2J} \rho_0(q)(\lambda - q)^{-1}\, dq. \tag{11}$$

Provided that λ lies a distance greater than $O(1/N)$ from the cut line $-2J < \lambda < 2J$ in the complex plane we can evaluate (11) and obtain

$$\langle G^0\rangle \equiv f(\lambda)/M_0 = (2J^2)^{-1}[\lambda - (\lambda - 4J^2)^{1/2}]. \tag{12}$$

For $M_0 > J$ there is a solution of the equation $f(\lambda) = 1$ at

$$\lambda = M_0 + J^2/M_0 \tag{13}$$

which is precisely the isolated eigenvalue obtained by Edwards and Jones (1976) and KTJ. The corresponding contribution to the averaged eigenvalue spectrum is easily seen from (10) to be $N^{-1}\delta[\lambda - (M_0 + J^2/M_0)]$. The consistency of our procedure can be checked by counting the number of states when λ lies in the range, $-2J < \lambda < 2J$, of the continuum of eigenvalues of the random matrix. This number is given by

$$-\frac{1}{\pi}\int_{-2J}^{+2J} d\lambda\, \mathrm{Im}\,\mathrm{Tr}\,\langle \mathbf{G}\rangle = -\frac{1}{\pi}\int_{-2J}^{+2J} d\lambda \left(\mathrm{Im}\,\mathrm{Tr}\,\langle \mathbf{G}^0\rangle + \frac{1}{2}\frac{\lambda - 2M_0}{(M_0 + J^2/M_0 - \lambda)(4J^2 - \lambda^2)^{1/2}}\right). \tag{14}$$

A straightforward contour integration shows this to be $N - 1$ for $M_0 > J$ and N when $M_0 < J$: i.e. as M_0 increases from zero across the value J, one state moves outside the range $|\lambda| < 2J$. This agrees with the eigenvalue density (2).

Equations (10) and (12) can also be used to obtain the moments of the eigenvalue density: the pth moment is the coefficient of $\lambda^{-(p+1)}$ in the expansion of $\mathrm{Tr}\,\langle \mathbf{G}\rangle$. The contribution arising from the matrix $\mathbf{M}_0$ is given by

$$\mathrm{Tr}\,\langle \mathbf{G}\rangle - \mathrm{Tr}\,\langle \mathbf{G}^0\rangle = \left(1 - \frac{4J^2}{\lambda^2}\right)^{-1/2} \sum_{n=1}^{\infty} \frac{M_0^n}{\lambda^{n+1}} 2^n \left[1 + \left(1 - \frac{4J^2}{\lambda^2}\right)^{1/2}\right]^{-n}. \tag{15}$$

The properties of the hypergeometric function $F(a, b, c; z)$ (Abramowitz and Stegun 1965, p 556) allow this to be written as

$$\begin{aligned}\mathrm{Tr}\,\langle \mathbf{G}\rangle - \mathrm{Tr}\,\langle \mathbf{G}^0\rangle &= \sum_{n=1}^{\infty} \frac{M_0^n}{\lambda^{n+1}} F(1 + \tfrac{1}{2}n, \tfrac{1}{2} + \tfrac{1}{2}n, 1 + n; 4J^2/\lambda^2) \\ &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{M_0^n J^{2m}}{\lambda^{n+2m+1}}\, {}^{n+2m}C_m.\end{aligned} \tag{16}$$

The error in the work of Kota and Potbhare (1977) occurs in their equation (2). Using the notation of the present Letter they write

$$(\mathbf{M}_0+\mathbf{J})^p = \mathbf{M}_0^p + \sum_{q=1}^{p-1} {}^pC_q(\mathbf{M}_0)^{p-q}\mathbf{J}^q + \mathbf{J}^p. \tag{17}$$

The matrices $\mathbf{M}_0$ and $\mathbf{J}$ do not however commute, and so the 'identity' (17) which the authors use as their starting point is false, and the rest of the Letter is in error. Because of the cyclic invariance of the trace this error does not show up until the fourth moment. There we have

$$\langle \mathrm{Tr}\,(\mathbf{M}_0+\mathbf{J})^4\rangle = \langle \mathrm{Tr}\,\mathbf{M}_0^4\rangle + 4\langle \mathrm{Tr}\,\mathbf{M}_0^2J^2\rangle + 2\langle \mathrm{Tr}\,\mathbf{M}_0\mathbf{J}\mathbf{M}_0\mathbf{J}\rangle + \langle \mathrm{Tr}\,\mathbf{J}^4\rangle. \tag{18}$$

We may evaluate this by using the statistical properties of the $u_\alpha(i)$ and dropping terms whose expectation value is zero. Direct evaluation of the second term on the right gives

$$\langle \mathrm{Tr}\,\mathbf{M}_0^2\mathbf{J}^2\rangle = M_0^2J^2 \tag{19}$$

whilst the third term gives

$$\langle \mathrm{Tr}\,\mathbf{M}_0\mathbf{J}\mathbf{M}_0\mathbf{J}\rangle = 2N^{-1}M_0^2J^2. \tag{20}$$

The mistake of Kota and Potbhare is to replace equation (20) by equation (19) so that the coefficient of $M_0^2J^2$ is given as 6 instead of 4. In general they find the coefficient of $M_0^{p-2q}J^{2q}$ to be $p!/(p-2q)!(q+1)!q!$ instead of the $p!/(p-q)!q!$ given by (16) as a result of their failure to take account of the fact that the matrices do not commute.

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